SYMMETRIC CHAIN DECOMPOSITIONS OF QUOTIENTS OF CHAIN PRODUCTS BY WREATH PRODUCTS

DWIGHT DUFFUS AND KYLE THAYER

Abstract. Subgroups of the symmetric group S_n act on powers of chains C^n by permuting coordinates, and induce automorphisms of the ordered sets C^n. The quotients defined are candidates for symmetric chain decompositions. We establish this for some families of groups in order to enlarge the collection of subgroups G of the symmetric group S_n for which the quotient B_n/G obtained from the G-orbits on the Boolean lattice B_n is a symmetric chain order.

1. Introduction

We are concerned with one of the most well-studied symmetry properties of a finite partially ordered set – possessing a partition into symmetric chains – and determining circumstances under which it is preserved by quotients.

A finite ordered set P with minimum element 0_P in which all maximal chains have the same length admits a rank function $r = r_P$, namely, for all $x \in P$, $r(x)$ is the maximum of all lengths $l(C) = |C|-1$ over all chains $C \subseteq P$ with minimum element 0_P and maximum element x. We let the *rank* $r(P)$ of P be the maximum value of $r(x)$ over all $x \in P$.

In a ranked order P the chain $x_1 < x_2 < \cdots < x_k$ is *saturated* if for each i there is no z such that $x_i < z < x_{i+1}$. Call the saturated chain *symmetric* if $r(x_1)+r(x_k) = r(P)$. A *symmetric chain decomposition* or *SCD* of P is a partition of P into symmetric chains. If P has an SCD, call P a *symmetric chain order*, or an *SCO*. Here, we are concerned with a particular family of SCOs, products of finite chains, and whether quotients of these are also SCOs.

For any partially ordered set P and any subgroup G of $\mathrm{Aut}(P)$ the automorphism group of P, let P/G denote the quotient poset. That is, the elements of P/G are the orbits induced by G and $[x] \leq [y]$ in P/G if there are $x' \in [x]$ and $y' \in [y]$ such that $x' \leq y'$ in P. Canfield and Mason conjectured [3] that for $P = B_n$, the Boolean lattice of all subsets of an n-element set ordered by containment, B_n/G is an SCO for all subgroups G of $\mathrm{Aut}(B_n) \cong S_n$, the symmetric group on $[n] := \{1, 2, \ldots, n\}$.

2000 *Mathematics Subject Classification.* Primary: 06A07.
Key words and phrases. symmetric chain decomposition; Boolean lattice.

A more general problem is to investigate conditions on an SCO P and a subgroup G of $\mathrm{Aut}(P)$ under which P/G an SCO.

In studying Venn diagrams, Griggs, Killian and Savage [13] explicitly constructed an SCD of the quotient B_n/G for n prime and given that G is generated by a single n-cycle. They asked if this *necklace* poset is an SCO for arbitrary n. Jordan [14] proved that it is by constructing an SCD of the quotient on the SCD in B_n based on an SCD obtained by Greene and Kleitman [10] for the Boolean lattice. P. Hersh and A. Schilling give another proof of Jordan's result with an explicit construction of an SCD in $B_n/\mathbb{Z}_n$ based on a cyclic version of Greene and Kleitman's bracketing procedure [12]. We showed that B_n/G is an SCO provided that G is generated by powers of disjoint cycles [8]. While this generalizes Jordan's result somewhat, the method of proof is likely more interesting in that we construct an SCD of the quotient by refining, or pruning, the Greene-Kleitman SCD in a more direct way.

Results can be extended in several ways. Dhand [5] has shown that if P is an SCO and $\mathbb{Z}_n$ acts on P^n in the usual way (by permuting coordinates) then $P^n/\mathbb{Z}_n$ is an SCO. (This is the same as considering P^n/G where G is generated by an n-cycle.) His methods are algebraic. As a special case, in some sense the base case of his argument, for any chain C, C^n/G is an SCO for G generated by an n-cycle. We have a slightly generalized version of this special case, that C^n/G is an SCO for G generated by powers of disjoint cycles [8]. The proof depends on showing that the Greene-Kleitman SCD can be pruned to give an SCD of the quotient.

Quotients of the form $(C_1 \times C_2 \times \cdots \times C_n)/K$, where each C_i is a chain, are interesting for a several reasons. First, it follows from a result of Stanley [18] that for any chain product and all subgroups K of its automorphism group, $(C_1 \times C_2 \times \cdots \times C_n)/K$ is rank-symmetric, rank-unimodal, and strongly Sperner (see, for instance, [14] for definitions). An SCO has these three properties. While these conditions are not sufficient to insure an SCD, Griggs [11] showed that a ranked ordered set with the LYM property, rank-symmetry and rank-unimodality is an SCO. And it is the case that products of chains have the LYM property. (We shall see, at the end of Section 3, that Stanley [16] was first to ask if certain quotients of B_n are SCOs.)

Second, the automorphism group of a product of chains consists of just those maps that permute coordinates corresponding to chains of the same length. To be more precise, for

$$P = \prod_{i=1}^{m} C_i^{n_i},\ C_j \not\cong C_k \text{ for } j \neq k,$$

and $\phi \in \mathrm{Aut}(P)$ there exist $\phi_i \in \mathrm{Aut}(C_i^{n_i})$ and $\sigma_i \in S_{n_i}$, $i = 1, 2, \ldots, m$, such that $\phi = (\phi_1, \phi_2, \ldots, \phi_m)$ and for $\mathbf{x} = (x_1, x_2, \ldots, x_{n_i}) \in C_i^{n_i}$,

$$\phi_i(\mathbf{x}) = (x_{\sigma_i^{-1}(1)}, x_{\sigma_i^{-1}(2)}, \ldots, x_{\sigma_i^{-1}(n_i)}).$$

In particular, $\mathrm{Aut}(P)$ is indentified with a subgroup of $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}$. These observations are justified by examining the action of an automorphism of

a chain product on the atoms of the product. They are also a consequence of results of Chang, Jónsson and Tarski on the strict refinement property for product decompositions of partially ordered sets [4].

Also, since a product of SCOs is an SCO, whenever a subgroup K of $\mathrm{Aut}(P)$ factors into subgroups K_i of $\mathrm{Aut}(C_i^{n_i})$, P/K is an SCO provided that each $C_i^{n_i}/K_i$ is an SCO. Thus, P/K is an SCO whenever each of the factors K_i is generated by powers of disjoint cycles, as noted in [8] as Corollary 1.

Finally, one can use the fact that $(C_1 \times C_2 \times \cdots \times C_n)/K$ has an SCD for certain groups K in order to prove that B_n/G has an SCD for groups G that are wreath products involving these K. This is described precisely in Section 2. The main results are stated and proved there as well. Examples and questions are presented in Section 3.

2. The Setup and Main Result

The observations about automorphisms of chain products can be stated in terms of wreath products of permutation groups. (See [6] for background on permutation groups.) For convenience, we identify $\mathrm{Aut}(B_n)$ with the symmetric group S_n.

Let $n = kt$ for $n, k, t \in \mathbb{N}$ and partition $[n]$ into t blocks, each of k consecutive members of $[n]$ – say

$$[n] = N_1 \cup N_2 \cup \cdots \cup N_t \quad \text{where} \quad N_r = [(r-1)k+1, \ldots, rk].$$

We consider those $\phi \in S_n$ for which there exist $\overline{\rho} = (\rho_1, \rho_2, \ldots, \rho_t)$, each $\rho_r \in S_k$, and $\tau \in S_t$ such that for each $r \in [t]$ and $i \in [k]$

$$\phi((r-1)k+i) = (\tau(r)-1)k + \rho_r(i). \tag{1}$$

Given subgroups K of S_k and T of S_t, the set G of all such $\phi \in S_n$ obtained from all $\overline{\rho} \in K^t$ and $\tau \in T$, is a subgroup of S_n, a wreath product, denoted by $G = K \wr T$.

Theorem 1. *Let n, k, t be positive integers and $n = kt$. Then B_n/G is a symmetric chain order for any subgroup G of $\mathrm{Aut}(B_n)$ defined as follows: $G = K \wr T$ where K is a subgroup of S_k, T is a subgroup of S_t, and both K and T are generated by powers of disjoint cycles.*

Let us assume that G is a subgroup of S_n and $G = K \wr T$, as described above. To see that the conditions on K and T in Theorem 1 ensure that B_n/G is an SCO, we use two steps to obtain the required description of B_n/G. First, we require a little terminology.

Given a partially ordered set P and subsets P_i, $i = 1, 2, \ldots, m$, use $P = \sum_{i=1}^m P_i$ to mean that P is partitioned by the family of P_i's, the order on P_i is the restriction of the order of P to P_i, with no information given about the order relations between elements of distinct P_i's. For a subposet Q of a ranked partially ordered set P, Q

is said to be *saturated* if for all $x, y \in Q$, x is covered by y in Q implies that x is covered by y in P. Given a saturated $Q \subseteq P$ with minimum 0_Q and maximum 1_Q, say that Q is *symmetric* in P if $r_P(0_Q) + r_P(1_Q) = r_P(P)$.

For $X \subset [n]$, let $X_r = X \cap N_r$, $r = 1, 2, \ldots, t$, and for any $q \in [t]$, let $X_{r,q} \subseteq N_q$ be given by

$$X_{r,q} = (q - r)k + X_r = \{(q - r)k + x \mid x \in X_r\}.$$

Thus, X_r and $X_{r,q}$ are translations of the same subset of $[k]$.

First, we consider the subgroup K' of G consisting of all ϕ for which $\tau = 1$. This is the base group of the wreath product and is isomorphic to K^t. Let K_r denote the copy of K acting on N_r and for any set N, let $B(N)$ denote the Boolean lattice of all subsets of N. Then

$$B_n/K' \;\cong\; \prod_{r=1}^{t} B(N_r)/K_r \;\cong\; (B_k/K)^t \tag{2}$$

The first isomorphism in (2) is given by the map $[A] \to ([A_1], [A_2], \ldots, [A_t])$, where and $[A]$ and $[A_r]$ denote the orbit of A under K' and the orbit of A_r under K_r, respectively.

Condition 1: B_k/K has an SCD.

Suppose that $C_1, C_2, \ldots, C_s$ constitute an SCD of B_k/K and that $C_1^r, C_2^r, \ldots, C_s^r$ is the corresponding SCD of $B(N_r)/K_r$, $r = 1, 2, \ldots, t$. Then these SCDs define a partition of the product $\prod_{r=1}^{t} B(N_r)/K_r$ into *grids*, that is, products of chains:

$$\prod_{r=1}^{t} B(N_r)/K_r \;=\; \prod_{r=1}^{t}\left(\sum_{j=1}^{s} C_j^r\right) \;=\; \sum_{\overline{j}=(j_1,j_2,\ldots,j_t)\in[s]^t} C_{j_1}^1 \times C_{j_2}^2 \ldots \times C_{j_t}^t. \tag{3}$$

Note that each grid $C(\overline{j}) = C_{j_1}^1 \times C_{j_2}^2 \ldots \times C_{j_t}^t$ is itself an SCO ([2], cf. [9])and is a symmetric, saturated subset of $\prod_{r=1}^{t} B(N_r)/K_r$. The family of all chains from the SCDs of the $C(\overline{j})$s and the first isomorphism in (2) yield an SCD for B_n/K'.

Since K' is a subgroup of G, the set of orbits defined by K' refines the set defined by G. We need to make a selection of exactly one K'-orbit from each G-orbit. This is done by selecting a family of grids from the partition of B_n/K' given in (3), and then selecting a subposet of each grid that inherits the SCD from the full grid. In the end, we obtain a system of distinct representatives for the G-orbits and an SCD of that system.

Consider the action of T on $[s]^t$, that is, for $\tau \in T$ and $\overline{j} = (j_1, j_2, \ldots, j_t) \in [s]^t$, $\tau(\overline{j}) = (j_{\tau^{-1}(1)}, j_{\tau^{-1}(2)}, \ldots, j_{\tau^{-1}(t)})$. Let $\widehat{\tau} : C(\overline{j}) \to C(\tau(\overline{j}))$ be defined by

$$\widehat{\tau}([X_1], [X_2], \ldots, [X_t]) = ([X_{\tau^{-1}(1),1}], [X_{\tau^{-1}(2),2}], \ldots, [X_{\tau^{-1}(t),t}]). \tag{4}$$

Then $\widehat{\tau}$ is well-defined and is an order isomorphism - both are easy to check. With $X = \cup_{r=1}^{t} X_r$ and $X' = \cup_{r=1}^{t} X_{\tau^{-1}(r),r}$, we can see that X and X' are in the same G-orbit because the map ϕ defined by $\phi((r-1)k+i) = (\tau(r)-1)k+i$ is in G [see (1)] and $\phi(X_r) = X_{r,\tau(r)}$, so $\phi(X) = X'$.

For $X \subseteq [n]$, let $\overline{X} = ([X_1],[X_2],\ldots,[X_t])$.

Claim: Let $X, Y \subseteq [n]$ with $\overline{X} \in C(\overline{j})$. Then $\phi(X) = Y$ for some $\phi \in G$ if and only if ϕ corresponds to $\overline{\rho} \in K^t$ and $\tau \in T$, as in (1), $\overline{Y} \in C(\tau(\overline{j}))$ and $\widehat{\tau}(\overline{X}) = \overline{Y}$.

Proof of Claim. By (4), $\overline{Y} \in C(\tau(\overline{j}))$ and $\widehat{\tau}(\overline{X}) = \overline{Y}$ if and only if $[Y_r] = [X_{\tau^{-1}(r),r}]$ for $r = 1,2,\ldots,t$.

And $[Y_r] = [X_{\tau^{-1}(r),r}]$ for each r if and only if there exists $\overline{\rho} = (\rho_1,\rho_2,\ldots,\rho_t) \in K^t$ such that

$$(\rho_{\tau^{-1}(r)}(X_{\tau^{-1}(r),1}))_{1,r} = Y_r \text{ for each } r. \tag{5}$$

With $\phi \in G$ corresponding to $\overline{\rho} \in K^t$ and $\tau \in T$, (1) shows that (5) holds if and only if

$$\phi(X_r) = (\rho_r(X_{r,1}))_{1,\tau(r)},\ \ r = 1,2,\ldots,t. \tag{6}$$

Finally, (6) holds if and only if

$$\phi(X) = \bigcup_{r=1}^{t} (\rho_{\tau^{-1}(r)}(X_{\tau^{-1}(r),1}))_{1,r} = \bigcup_{r=1}^{t} Y_r = Y. \quad \square$$

Select a representative from each orbit in $[s]^t$ under T, say the lexicographically least, and let J be the set of these representatives. The family $\{C(\overline{j}) \mid \overline{j} \in J\}$ seems to be a promising source of an SCD of B_n/G. Indeed, for $X = \cup_{r=1}^{t} X_r \subseteq [n]$, each X_r determines an index j_r via $[X_r] \in C_{j_r}^{r}$ in the SCD of $B(N_r)/K_r$. Thus, there is a unique $\overline{j} \in J$ with $([X_1],[X_2],\ldots,[X_t]) \in C(\overline{j})$.

Now let us suppose that X and Y satisfy the conditions of the Claim. Then $\overline{X}, \overline{Y} \in C(\overline{j})$ for some $\overline{j} \in J$ if and only if $\tau(\overline{j}) = \overline{j}$, that is, exactly if $\tau \in T_{\overline{j}}$, the pointwise stabilizer of $\overline{j}$ in T acting on $[s]^t$. Therefore, the following condition is enough to ensure that $B(n)/G$ is an SCO.

Condition 2: With the SCD $C_1, C_2, \ldots, C_s$ of $B(k)/K$, for each $\overline{j} \in [s]^t$, $C(\overline{j})/T_{\overline{j}}$ is an SCO, where $T_{\overline{j}}$ is the stabilizer of $\overline{j}$ in T.

By Corollary 1 in [8], the quotient of a product of chains by a group of automorphisms generated by powers of disjoint cycles is an SCO. Thus, Theorem 1 follows from this fact.

Lemma 1. *Let s, t be positive integers and let T be a subgroup of S_t that is generated by powers of disjoint cycles. Then for all $\overline{j} \in [s]^t$, the stabilizer $T_{\overline{j}}$ of $\overline{j}$ in the action of T on $[s]^t$ is also generated by powers of disjoint cycles.*

Proof of Lemma 1. Let $\overline{j} \in [s]^t$ and $\tau \in T$. Then $\tau \in T_{\overline{j}}$ if and only if for all $u, v \in [t]$, $\tau(u) = v$ implies that $j_u = j_v$. In other words, the partition of $[t]$ defined by the orbits of $[t]$ under τ must refine the partition of $[t]$ defined $\overline{j}$. For each $\tau \in T$ there is a minimum m such that $\tau^m \in T_{\overline{j}}$. This is because $\tau^k \in T_{\overline{j}}$ for at least one k, the order of τ, and if $\tau^a, \tau^b \in T_{\overline{j}}$ then with $d = \gcd(a, b)$, $\tau^d \in T_{\overline{j}}$.

Suppose that $T = \langle \sigma_1^{r_1}, \sigma_2^{r_2}, \ldots, \sigma_m^{r_m} \rangle$ and d_i is minimum such that $(\sigma_i^{r_i})^{d_i} \in T_{\overline{j}}$ for $i = 1, 2, \ldots, m$. Then $T_{\overline{j}} = \langle \sigma_1^{r_1 d_1}, \sigma_2^{r_2 d_2}, \ldots, \sigma_m^{r_m d_m} \rangle$. $\square$

This completes the proof of Theorem 1.

3. Examples and Questions

The subgroups of S_n to which Theorem 1 and the results of [8] apply are certainly quite special and form a restricted class. However, it is interesting to see a catalog of those subgroups of S_n that, by these results, do generate quotients of B_n with SCDs for some small values of n. We display the collection for $n = 6$.

First, let's review the facts we can apply. Let G be a subgroup of S_n.

(1) If $[n] = X_1 \cup X_2 \cup \cdots \cup X_m$ is a partition and $G = G_1 \times G_2 \times \cdots \times G_m$ where $G_i = G_{|X_i}$ then B_n/G is an SCO provided that each $B(X_i)/G_i$ is an SCO.

(2) If $G = \langle \sigma^r \rangle$ for some cycle $\sigma \in S_n$ then B_n/G is an SCO.

(3) Theorem 1.

For $n = 6$, we can organize the subgroups by listing them according to the partitions of 6 given by the orbit sizes, then refining with the cycle structure. The list given should be all the subgroups, up to conjugation, that are handled by **(1) - (3)** and the observation that the quotient of B_n by either the symmetric group S_n or the alternating group A_n is an $n+1$-element chain.

The partitions $6 = 1 + 1 + \cdots + 1$ and $6 = 1 + \cdots + 1 + 2$ each give only one group, (1) and $\mathbb{Z}_2 \cong \langle (12) \rangle$, respectively. Also, $6 = 1 + 1 + 1 + 3$ yields two: $\mathbb{Z}_3 \cong \langle (1\ 2\ 3) \rangle$ and $S_3 \cong \langle (1\ 2\ 3), (12) \rangle$. The rest are listed in this table.

Partition of 6	**Generators**	**Description**	**Order**
$1+1+2+2$	(1 2), (3 4)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4
	(1 2)(3 4)	$\mathbb{Z}_2$	2
$1+1+4$	(1 2 3 4)	$\mathbb{Z}_4$	4
	(1 2 3 4), (1 2)	S_4	24
	(1 2 3), (2 3 4)	A_4	12
	(1 2 3 4), (1 3)	$\mathbb{Z}_2 \wr \mathbb{Z}_2 \cong D_8$	8
$1+2+3$	(1 2), (3 4 5)	$\mathbb{Z}_2 \times \mathbb{Z}_3$	6
	(1 2), (3 4 5), (3 4)	$\mathbb{Z}_2 \times S_3$	12
$2+2+2$	(1 2), (3 4), (5 6)	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	8
	(1 2)(3 4), (5 6)	$\mathbb{Z}_2 \times \mathbb{Z}_2$	4
	(1 2)(3 4)(5 6)	$\mathbb{Z}_2$	2
$3+3$	(1 2 3), (4 5 6)	$\mathbb{Z}_3 \times \mathbb{Z}_3$	9
	(1 2 3), (1 2), (4 5 6)	$S_3 \times \mathbb{Z}_3$	18
	(1 2 3), (1 2), (4 5 6), (4 5)	$S_3 \times S_3$	36
$2+4$	(1 2), (3 4 5 6)	$\mathbb{Z}_2 \times \mathbb{Z}_4$	8
	(1 2), (3 4 5 6), (3 4)	$\mathbb{Z}_2 \times S_4$	48
	(1 2), (3 4 5), (4 5 6)	$\mathbb{Z}_2 \times A_4$	24
	(1 2), (3 4 5 6), (3 5)	$\mathbb{Z}_2 \times (\mathbb{Z}_2 \wr \mathbb{Z}_2)$	16
$1+5$	(1 2 3 4 5)	$\mathbb{Z}_5$	5
	(1 2 3 4 5), (1 2 3)	A_5	60
	(1 2 3 4 5), (1 2)	S_5	120
6	(1 2 3 4 5 6)	$\mathbb{Z}_6$	6
	(1 2 3 4 5), (1 2 3), (1 2)(5 6)	A_6	360
	(1 2 3 4 5 6), (1 2)	S_6	720
	(1 2 3)(4 5 6), (1 4)	$\mathbb{Z}_2 \wr \mathbb{Z}_3$	24
	(1 2 3), (1 4)(2 5)(3 6)	$\mathbb{Z}_3 \wr \mathbb{Z}_2$	18
	(1 2 3), (1 2), (1 4)(2 5)(3 6)	$S_3 \wr \mathbb{Z}_2$	72

The obvious next case to try to settle involves the dihedral groups.

Problem 1: Show that for all n, D_{2n}, the dihedral group of symmetries of the regular n-gon, satisfies: B_n/D_{2n} is an SCO.

We know this only for the trivial cases $n = 1, 2, 3$ and, as noted in the preceding table, $n = 4$, since D_8 is a wreath product $\mathbb{Z}_2 \wr \mathbb{Z}_2$. Note that D_{2n} is the semidirect product $\mathbb{Z}_n \rtimes \mathbb{Z}_2$, however the dihedral groups are not wreath products, for $n \geq 5$. This is easy to see: each permutation in D_{2n} has either no fixed points, or one (for odd n) or two (for even n), or n fixed points. However, a wreath product $K \wr T$ has maps with $k, 2k, \ldots, tk$ fixed points, where $K \leq S_k$ and $T \leq S_t$.

The approach presented above can be extended in some other special cases. For instance, the subgroup G of S_8 generated by $\{(15)(26)(37)(48), (14)(23)\}$ is a wreath product $K \wr T$ with $K \leq S_4$, generated by $(1243)^2$ and $T \leq S_2$ generated by (12). Thus B_8/G is an SCO. If we let H be the subgroup of G generated by $\{(15)(26)(37)(48), (14)(23)(58)(67)\}$ then one can describe how to refine the G-orbits to obtain H-orbits and how to produce the additional symmetric chains required for an SCD of B_8/H. Unfortunately, this sort of argument does not appear to yield anything for D_{2n} since it is not a "manageable" subgroup of any group we can handle, such as $Z_n \wr Z_2$ or $Z_2 \wr Z_n$.

As noted in the introduction, in [16] Stanley first raised the question of whether the distributive lattices $L(k, t)$, the set of all integer sequences $\mathbf{a} = (a_1, a_2, \ldots, a_t)$ such that $0 \leq a_1 \leq \cdots \leq a_t \leq k$ ordered componentwise, is an SCO. He also noted [17, 18] that $L(k, t)$ is obtained as a quotient of the Boolean lattice B_{kt} by the group $S_k \wr S_t$ with its natural action on $[n]$, with $n = kt$. (This is the action described at the beginning of Section 2.) It is easiest to see this if we think of $L(k, t)$ as the set of down sets of the grid $\mathbf{k} \times \mathbf{t}$ ordered by containment. The wreath product $S_k \wr S_t$ acts on the elements of $\mathbf{k} \times \mathbf{t}$ by permuting the elements of $C_j = \{(1, j), (2, j), \ldots, (k, j)\}$ freely for each $j = 1, 2, \ldots, t$, then permuting $C_1, C_2, \ldots, C_t$. Each equivalence class under this action contains a unique down set of $\mathbf{k} \times \mathbf{t}$.

If we regard this description in two steps, then the independent permutations of the elements in each $C_j, j = 1, 2, \ldots, t$, produces the quotient $(B_k/S_k)^t$, just as in (2) in Section 2. In this case, $B_k/S_k \cong \mathbf{k} + \mathbf{1}$, the $k + 1$-element chain. The second part of the action is just S_t acting on the coordinates of $(\mathbf{k} + \mathbf{1})^t$. We restate Stanley's problem.

Problem 2: Show that for all k, t, $L(k, t)$ is an SCO. Equivalently, show that the quotient

$$B_{kt}/(S_k \wr S_t) \;\cong\; (\mathbf{k} + \mathbf{1})^t/S_t$$

is an SCO.

References

[1] V. B. Alekseev, Use of symmetry in finding the width of partially ordered sets (Russian), *Diskret. Analiz.* **26** (1974) 20 - 35.

[2] N. G. de Bruijn, C. Tengbergen and D. Kruyswijk, On the set of divisors of a number, *Nieuw Arch. Wiskd.* **23** (1951) 191 - 193.

[3] E. R. Canfield and S. Mason, When is a quotient of the Boolean lattice a symmetric chain order?, preprint (2006).

[4] C. C. Chang, B. Jónsson and A. Tarski, Refinement properties for relational structures, *Fund. Math.* **55** (1964) 249 - 281.

[5] V. Dhand, Symmetric chain decomposition of necklace posets, newblock *Electron. J. Combin.*, **19** (2012) P26.

[6] John D. Dixon and Brian Mortimer, *Permuation Groups*, Springer, New York (1996).

[7] D. Duffus, P. Hanlon and R. Roth, Matchings and hamiltonian cycles in some families of symmetric graphs, *Emory University Technical Reports* (1986).

[8] D. Duffus, J. McKibben-Sanders and Kyle Thayer, Some quotients of the Boolean lattice are symmetric chain orders, *Electron. J. Combin.* (2012) **19(2)** P46 .

[9] K. Engel, *Sperner Theory*, Cambridge University Press, Cambridge UK (1997).

[10] C. Greene and D. J. Kleitman, Strong versions of Sperner's theorem, *J. Combinatorial Theory A* **20** (1976) 80 - 88.

[11] J. R. Griggs, Sufficient conditions for a symmetric chain order, *SIAM J. Appl. Math.* **32** (1977) 807 - 809.

[12] P. Hersh and A. Schilling. Symmetric chain decomposition for cyclic quotients of Boolean algebras and relation to cyclic crystals. *Internat. Math. Res. Notices* doi 10.1093/imrn/rnr254

[13] J. R. Griggs, C. E. Killian and C. D. Savage, Venn diagrams and symmetric chain decompositions in the Boolean lattice, *Electronic J. Combinatorics* **11** (2004) R2.

[14] K. K. Jordan, The necklace poset is a symmetric chain order, *J. Combinatorial Theory A* **117** (2010) 625 - 641.

[15] M. Pouzet and I. G. Rosenberg, Sperner properties for groups and relations, *Europ. J. Combinatorics* **7** (1986) 349 - 370.

[16] R. P. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property, *SIAM J. Alg. Disc. Meth.* **1** (1980) 168 - 184

[17] R. P. Stanley, Some aspects of groups acting on finite posets, *J. Combinatorial Theory A* **32** (1982) 132 - 161

[18] R. P. Stanley, Quotients of Peck posets, *Order* **1** (1984) 29 - 34.

Mathematics & Computer Science Department, Emory University, Atlanta, GA 30322, USA

E-mail address, Dwight Duffus: dwight@mathcs.emory.edu

Mathematics & Computer Science Department, Emory University, Atlanta, GA 30322, USA

E-mail address, Kyle Thayer: kyle.thayer@gmail.com